

Detecting Algebraic Manipulation in Leaky Storage Systems

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Abstract. Algebraic Manipulation Detection (AMD) Codes detect adversarial noise that is added to a coded message which is stored in a storage that is opaque to the adversary. We study AMD codes when the storage can leak up to $\rho \log |\mathcal{G}|$ bits of information about the stored codeword, where \mathcal{G} is the group that contains the codeword and ρ is a constant. We propose ρ -AMD codes that provide protection in this new setting. We define weak and strong ρ -AMD codes that provide security for a random and an arbitrary message, respectively. We derive concrete and asymptotic bounds for the efficiency of these codes featuring a rate upper bound of $1 - \rho$ for the strong codes. We also define the class of ρ^{LV} -AMD codes that provide protection when leakage is in the form of a number of codeword components, and give constructions featuring a family of strong ρ^{LV} -AMD codes that asymptotically achieve the rate $1 - \rho$. We describe applications of ρ -AMD codes to, (i) robust ramp secret sharing scheme and, (ii) wiretap II channel when the adversary can eavesdrop a ρ fraction of codeword components and tamper with all components of the codeword.

1 Introduction

Algebraic Manipulation Detection (AMD) Codes [1] protect messages against additive adversarial tampering, assuming the codeword cannot be “seen” by the adversary. In AMD codes, a message is encoded to a codeword that is an element of a publicly known group \mathcal{G} . The codeword is stored in a private storage which is perfectly opaque to the adversary. The adversary however can add an *arbitrary* element of \mathcal{G} to the storage to make the decoder output a different message. A δ -secure AMD code guarantees that any such manipulation succeeds with probability at most δ . Security of AMD codes has been defined for “weak” and “strong” codes: weak codes provide security assuming message distribution is uniform, while strong codes guarantee security for any message distribution. Weak AMD codes are primarily deterministic codes and security relies on the randomness of the message space. Strong AMD codes are randomized codes and provide security for any message. AMD codes have wide applications as a building block of cryptographic primitives such as robust information dispersal [1], and anonymous message transmission [1], and have been used to provide a

generic construction for robust secret sharing schemes from linear secret sharing schemes [1].

AMD codes with leakage were first considered in [2] where the leakage was defined for specific parts of the encoding process. An α -weak AMD code with linear leakage, also called α -weak LLR-AMD code, is a deterministic code that guarantees security when part of the message is leaked but the min-entropy of the message space is at least $1 - \alpha$ fraction of the message length (in bits). An α -strong LLR-AMD is a randomized code that guarantees security when the randomness of encoding, although partially leaked, has at least min-entropy $(1 - \alpha) \log |\mathcal{R}|$ where \mathcal{R} is the randomness set of encoding.

In this paper we consider leakage from the storage that holds the codeword. This effectively relaxes the original model of AMD codes that required the codeword to be perfectly private. As we will show this model turns out to be more challenging compared to LLR-AMD models where the leakage is in a more restricted part in the encoding process.

A more detailed relation between our model and LLR-AMD models is given in Section 3.1.

Our work

We define ρ -Algebraic Manipulation Detection (ρ -AMD) codes as an extension of AMD codes when the storage that holds the codeword (an element of \mathcal{G}), leaks up to $\rho \log |\mathcal{G}|$ bits of information about the codeword. We assume the adversary can apply an arbitrary function to the storage and receive up to $\rho \log |\mathcal{G}|$ bits of information about the codeword. Similar to the original AMD codes, we define weak and strong ρ -AMD codes as deterministic and randomized codes that guarantee security for a uniformly distributed message and any message, respectively.

Efficiency of ρ -AMD codes is defined concretely (similar to [1]) and asymptotically (using the *rate of the code family*, which is the asymptotic ratio of the message length to the codeword length, as the message length approaches infinity). We prove concrete bounds for both strong and weak ρ -AMD codes and a non-trivial upper bound $1 - \rho$ on the rate of the strong ρ -AMD codes. Comparison of bounds for different models of AMD codes is summarized in Table 1.

For construction, we use the relationship between ρ -AMD codes and LLR-AMD codes, to construct (non-optimal) ρ -AMD codes, and leave the construction of rate optimal ρ -AMD codes as an interesting open problem. We however define a special type of leakage in which leakage is specified by the number of codeword components that the adversary can select for eavesdropping. The model is called *limited-view ρ -AMD* (ρ^{LV} -AMD). The ρ^{LV} -AMD adversary is allowed to select a fraction ρ of the *codeword components*, and select their tampering (offset) vector after seeing the values of the chosen components. This definition of limited-view adversary was first used in [3] where the writing power of the adversary was also parametrized. We give an explicit construction of strong

codes	concrete bound	rate bound
strong AMD	$G \geq \frac{M-1}{\delta^2} + 1$	1
strong ρ-AMD	$\mathbf{G}^{1-\rho} \geq \frac{M-1}{\delta^2} + 1$	$1 - \rho$
α -strong LLR-AMD	$G \geq \frac{(M-1)(1-e^{-1})}{\delta^2} + 1$	1
weak AMD	$G \geq \frac{M-1}{\delta} + 1$	1
weak ρ-AMD	$\mathbf{G} \geq \frac{M-1}{\delta} + 1$ and $\mathbf{M} \geq \frac{G^\rho}{\delta}$	$\mathbf{1}$
α -weak LLR-AMD	$G \geq \frac{(M-1)(1-e^{-1})}{\delta^{1-\alpha}} + 1$ and $G \geq \frac{M^\alpha(M-1)(1-e^{-1})}{\delta} + 1$	$\frac{1}{1+\alpha}$

Table 1. G denotes the size of the group \mathcal{G} that codewords live in and M denotes the size of the message set \mathcal{M} . δ is the security parameter.

ρ^{LV} -AMD codes that achieve rate $1 - \rho$, using an AMD code and a wiretap II code as building blocks. We note that this rate is achievable for large constant size alphabets, if we allow a seeded encoder involving a universal hash family (see [15]). That is the alphabet size depends on the closeness to the actual capacity value. Also we do not know if $1 - \rho$ is the capacity of strong ρ^{LV} -AMD codes. Finding the capacity of strong ρ^{LV} -AMD codes however is an open question as the type of leakage (component wise) is more restricted than strong ρ -AMD codes. We also construct a family of weak ρ^{LV} -AMD codes that achieve rate 1 for any leakage parameter ρ .

We consider two applications. The first application can be seen as parallel to the application of the original AMD codes to robust secret sharing scheme. The second application is a new variation of active adversary wiretap channel II.

Robust ramp secret sharing scheme. A (t, r, N) -ramp secret sharing scheme [7,13] is a secret sharing scheme with two thresholds, t and r , such that any t or less shares do not leak any information about the secret while any r or more shares reconstruct the secret and if the number a of shares is in between t and r , an $\frac{a-t}{r-t}$ fraction of information of the secret will be leaked. We define a *robust ramp secret sharing scheme* as a ramp secret sharing scheme with an additional (ρ, δ) -robustness property which requires that the probability of reconstructing a wrong secret, if up to $t + \lfloor \rho(r-t) \rfloor$ shares are controlled by an active adversary, is bounded by δ . Here ρ is a constant. We will show that a (t, r, N, ρ, δ) -robust secret sharing scheme can be constructed from a linear (t, r, N) -ramp secret sharing scheme, by first encoding the message using a ρ -AMD code with security parameter δ , and then using the linear ramp secret sharing scheme to generate shares.

Wiretap II with an algebraic manipulation adversary. Wiretap model of communication was proposed by Wyner [4]. In wiretap II setting [5], the goal is to provide secrecy against a passive adversary who can adaptively select a fraction ρ of transmitted codeword components to eavesdrop. We consider active wiretap II adversaries that in addition to eavesdropping the channel, algebraically manipulate the communication by adding a noise (offset) vector to the sent codeword. The code must protect against eavesdropping and also detect tampering. An algebraic manipulation wiretap II code is a wiretap II code with security

against an eavesdropping adversary and so the rate upper bound for wiretap II codes is applicable. Our construction of ρ^{LV} -AMD codes gives a family of algebraic manipulation wiretap II codes which achieve this rate upper bound and so the construction is capacity-achieving. The result effectively shows that algebraic manipulation detection in this case can be achieved for “free” (without rate loss), asymptotically

Table 2 summarizes the code constructions and applications.

codes constructed	asymptotic rate	applications
strong ρ-AMD strong ρ^{LV}-AMD	N.A. $1 - \rho$	(ρ, δ) - robust ramp secret sharing $(\rho, 0, \delta)$ - algebraic adversary wiretap II
weak ρ-AMD weak ρ^{LV}-AMD	N.A. 1	N.A. N.A.

Table 2. Summary of codes constructed in this paper and their applications.

Related works

Related works. AMD codes were proposed in [1] and have found numerous applications. A work directly comparable to ours is [2] where LLR-AMD code with different leakage models for weak and strong codes are introduced. Our leakage model uses a single leakage model for both weak and strong codes and is a natural generalization of the original AMD codes. The relation between our model and LLR-AMD codes is given in Section 3.1. More generally, there is a large body of work on modelling leakage and designing leakage resilient systems. A survey can be found in [6].

Ramp secret sharing schemes (ramp SSS) are introduced in [7]. Robust secret sharing schemes (robust SSS) are well studied (see for example [1]). To our knowledge robust ramp secret sharing schemes (robust ramp SSS) have not been considered before. In a robust SSS, robustness is defined only when the number of the compromised players is below the privacy threshold of the underlying SSS. Our definition of robust ramp SSS has robustness guarantee even when the number of compromised players is bigger than the privacy threshold.

Wiretap II model with active adversary was first studied in [14], where the eavesdropped components and tampered components are restricted to be the same set. A general model of wiretap II adversaries with additive manipulation was defined in [8]. In this model (called adversarial wiretap or AWTP) the adversary can read a fraction ρ_r , and add noise to a fraction ρ_w , of the codeword components. The goal of the encoding scheme is to provide secrecy and guarantee reliability (message recovery) against this adversary. A variation of AWTP called eAWTP is studied in [15], where erasure of codeword components instead of additive tampering is considered. Interestingly, both AWTP and eAWTP have the same capacity $1 - \rho_r - \rho_w$. The alphabet of known capacity-achieving codes

are, $\mathcal{O}(\frac{1}{\xi^4 \epsilon^2})$ for AWTP codes and $\mathcal{O}(2^{\frac{1}{\epsilon^2}})$ for eAWTP codes, respectively, where ξ is the difference of the actual rate and capacity [15]. The adversary of algebraic manipulation wiretap II codes defined in this paper can be seen as the AWTP adversary with $\rho_r = \rho$ and $\rho_w = 1$, yielding $1 - \rho_r - \rho_w < 0$. In this case recovering the message is impossible. Our results on algebraic manipulation wiretap II show that a weaker goal against active attack, that is *to detect* manipulation of the message, is achievable and can be achieved with capacity $1 - \rho$, which is the same as the capacity of wiretap II codes with no security against active attacks.

Organization: In Section 2, we give notations and introduce AMD codes (with/without leakage) and wiretap II codes. In Section 3, we define ρ -AMD codes and derive efficiency bounds. In Section 4, we study ρ^{LV} -AMD codes and give concrete constructions. In Section 5, we give two applications.

2 Preliminaries

Calligraphy letters \mathcal{X} denote sets and their corresponding capital letters denote the cardinality, $|\mathcal{X}| = X$. Boldface letters \mathbf{x} denote vectors. $\mathbf{x}_{|S}$ denotes the sub-vector of \mathbf{x} consisting of the components specified by the index set S . $[n]$ denotes $\{1, 2, \dots, n\}$. Capital boldface letters \mathbf{X} denote random variables, and $\mathbf{X} \leftarrow \mathcal{X}$ denotes sampling of the random variable \mathbf{X} from the set \mathcal{X} , with $\mathbf{X} \stackrel{\$}{\leftarrow} \mathcal{X}$ denoting a uniform distribution in sampling. The statistical distance between \mathbf{X} and \mathbf{Y} that are both defined over the set \mathcal{W} , is defined as,

$$\text{SD}(\mathbf{X}, \mathbf{Y}) \triangleq \frac{1}{2} \sum_{\mathbf{w} \in \mathcal{W}} |\Pr[\mathbf{X} = \mathbf{w}] - \Pr[\mathbf{Y} = \mathbf{w}]|.$$

We say \mathbf{X} and \mathbf{Y} are δ -close if $\text{SD}(\mathbf{X}, \mathbf{Y}) \leq \delta$. The *min-entropy* $H_\infty(\mathbf{X})$ of a random variable $\mathbf{X} \leftarrow \mathcal{X}$ is

$$H_\infty(\mathbf{X}) = -\log \max_{\mathbf{x} \in \mathcal{X}} \Pr[\mathbf{X} = \mathbf{x}].$$

The (average) *conditional min-entropy* $\tilde{H}_\infty(\mathbf{X}|\mathbf{Z})$ of \mathbf{X} conditioned on \mathbf{Z} is defined [9] as,

$$\tilde{H}_\infty(\mathbf{X}|\mathbf{Z}) = -\log \left(\mathbb{E}_{\mathbf{Z}=\mathbf{z}} \max_{\mathbf{x}} \Pr[\mathbf{X} = \mathbf{x}|\mathbf{Z} = \mathbf{z}] \right).$$

The following bound on the amount of information about one variable that can leak through a correlated variable is proved in [9].

Lemma 1. [9] Let $\mathbf{X} \leftarrow \mathcal{X}$ and $\mathbf{Z} \leftarrow \mathcal{Z}$ with $\ell = \log |\mathcal{Z}|$. Then

$$\tilde{H}_\infty(\mathbf{X}|\mathbf{Z}) \geq H_\infty(\mathbf{X}) - \ell.$$

Definition 1. An (M, G, δ) -algebraic manipulation detection code, or (M, G, δ) -AMD code for short, is a probabilistic encoding map $Enc : \mathcal{M} \rightarrow \mathcal{G}$ from a set \mathcal{M} of size M to an (additive) group \mathcal{G} of order G , together with a deterministic decoding function $Dec : \mathcal{G} \rightarrow \mathcal{M} \cup \{\perp\}$ such that $Dec(Enc(\mathbf{m})) = \mathbf{m}$ with probability 1 for any $\mathbf{m} \in \mathcal{M}$. The security of an AMD code requires that for any $\mathbf{m} \in \mathcal{M}$, $\Delta \in \mathcal{G}$, $\Pr[Dec(Enc(\mathbf{m}) + \Delta) \notin \{\mathbf{m}, \perp\}] \leq \delta$.

The AMD code above is said to provide *strong security*. Weak AMD codes provide security for randomly chosen messages. Efficiency of (M, G, δ) -AMD codes is measured by the *effective tag size* which is defined as the minimum tag length $\min\{\log_2 G\} - u$, where the minimum is over all (M, G, δ) -AMD codes with $M \geq 2^u$. Concrete lengths are important in practice, and additionally, the asymptotic rate (defined as the limit of the ratio of message length to codeword length as the length grows to infinity) of both weak and strong AMD codes has been shown [1] to be 1.

Lemma 2. [1] Any weak, respectively strong, (M, G, δ) -AMD code satisfies

$$G \geq \frac{M-1}{\delta} + 1, \text{ respectively, } G \geq \frac{M-1}{\delta^2} + 1.$$

The following construction is optimal with respect to effective tag size.

Construction 1 [1]: Let \mathbb{F}_q be a field of size q and characteristic p , and let d be any integer such that $d+2$ is not divisible by p . Define the encoding function,

$$Enc : \mathbb{F}_q^d \rightarrow \mathbb{F}_q^d \times \mathbb{F}_q \times \mathbb{F}_q, \mathbf{m} \mapsto (\mathbf{m}, \mathbf{r}, f(\mathbf{r}, \mathbf{m})), \text{ where } f(\mathbf{r}, \mathbf{m}) = \mathbf{r}^{d+2} + \sum_{i=1}^d m_i \mathbf{r}^i.$$

The decoder Dec verifies a tagged message $(\mathbf{m}, \mathbf{r}, t)$ by comparing $t = f(\mathbf{r}, \mathbf{m})$ and outputs \mathbf{m} if agree; \perp otherwise. (Enc, Dec) gives a $(q^d, q^{d+2}, \frac{d+1}{q})$ -AMD code.

Definition 2 (strong LLR-AMD). [2] A randomized code with encoding function $Enc : \mathcal{M} \times \mathcal{R} \rightarrow \mathcal{X}$ and decoding function $Dec : \mathcal{X} \rightarrow \mathcal{M} \cup \{\perp\}$ is a $(M, X, |\mathcal{R}|, \alpha, \delta)$ -strong LLR-AMD code if for any $\mathbf{m} \in \mathcal{M}$ and any $\mathbf{r} \in \mathcal{R}$, $Dec(Enc(\mathbf{m}, \mathbf{r})) = \mathbf{m}$, and for any adversary \mathbb{A} and variables $\mathbf{R} \xleftarrow{\$} \mathcal{R}$ and \mathbf{Z} such that $\tilde{H}_\infty(\mathbf{R}|\mathbf{Z}) \geq (1 - \alpha) \log |\mathcal{R}|$, it holds for any $\mathbf{m} \in \mathcal{M}$:

$$\Pr[Dec(Enc(\mathbf{m}, \mathbf{R}) + \mathbb{A}(\mathbf{Z})) \notin \{\mathbf{m}, \perp\}] \leq \delta, \quad (1)$$

where the probability is over the randomness of encoding.

Definition 3 (weak LLR-AMD). [2] A deterministic code with encoding function $Enc : \mathcal{M} \rightarrow \mathcal{X}$ and decoding function $Dec : \mathcal{X} \rightarrow \mathcal{M} \cup \{\perp\}$ is a (M, X, α, δ) -weak LLR-AMD code if for any $\mathbf{m} \in \mathcal{M}$, $Dec(Enc(\mathbf{m})) = \mathbf{m}$, and for any adversary \mathbb{A} and variables $\mathbf{M} \leftarrow \mathcal{M}$ and \mathbf{Z} such that $\tilde{H}_\infty(\mathbf{M}|\mathbf{Z}) \geq (1 - \alpha) \log |\mathcal{M}|$, it holds:

$$\Pr[Dec(Enc(\mathbf{M}) + \mathbb{A}(\mathbf{Z})) \notin \{\mathbf{M}, \perp\}] \leq \delta, \quad (2)$$

where the probability is over the randomness of the message.

In the above two definitions, leakages are from randomness (bounded by $\tilde{H}_\infty(\mathbf{R}|\mathbf{Z}) \geq (1 - \alpha) \log |\mathcal{R}|$) and message space (bounded by $\tilde{H}_\infty(\mathbf{M}|\mathbf{Z}) \geq (1 - \alpha) \log |\mathcal{M}|$), respectively.

Wiretap II codes.

Wiretap II model [5] of secure communication considers a scenario where Alice wants to send messages to Bob over a reliable channel that is eavesdropped by an adversary, Eve. The adversary can read a fraction ρ of the transmitted codeword components, and is allowed to choose any subset (the right size) of their choice. A wiretap II code provides information-theoretic secrecy for message transmission against this adversary.

Definition 4. A (ρ, ε) wiretap II code, or (ρ, ε) -WtII code for short, is a probabilistic encoding function $Enc : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^n$, together with a deterministic decoding function $Dec : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^k$ such that $Dec(Enc(\mathbf{m})) = \mathbf{m}$ for any $\mathbf{m} \in \mathbb{F}_q^k$. The security of a (ρ, ε) -WtII code requires that for any $\mathbf{m}_0, \mathbf{m}_1 \in \mathbb{F}_q^k$, any $S \subset [n]$ of size $|S| \leq n\rho$,

$$SD(Enc(\mathbf{m}_0)_{|S}; Enc(\mathbf{m}_1)_{|S}) \leq \varepsilon \quad (3)$$

A rate R is achievable if there exists a family of (ρ, ε) -WtII codes with encoding and decoding functions $\{Enc_n, Dec_n\}$ such that $\lim_{n \rightarrow \infty} \frac{k}{n} = R$.

The above definition of security is in line with [8] and is stronger than the original definition [5], and also the definition in [10].

Lemma 3. [8] The achievable rate of $(\rho, 0)$ -WtII codes is upper bounded by $1 - \rho$.

When $\varepsilon = 0$ is achieved in (3), the distribution of any ρ fraction of the codeword components is independent of the message. This is achieved, for example, by the following construction of wiretap II codes.

Construction 2 [5] Let $G_{(n-k) \times n}$ be a generator matrix of a $[n, n - k]$ MDS code over \mathbb{F}_q . Append k rows to G such that the obtained matrix $\begin{bmatrix} G \\ \tilde{G} \end{bmatrix}$ is non-singular. Define the encoder $WtIenc$ as follows.

$$WtIenc(\mathbf{m}) = [\mathbf{r}, \mathbf{m}] \begin{bmatrix} G \\ \tilde{G} \end{bmatrix}, \text{ where } \mathbf{r} \xleftarrow{\$} \mathbb{F}_q^{n-k}.$$

$WtIdec$ uses a parity-check matrix $H_{k \times n}$ of the MDS code to first compute the syndrome, $H\mathbf{x}^T$, and then map the syndrome back to the message using the one-to-one correspondence between syndromes and messages. The above construction gives a family of $(\rho, 0)$ -WtII codes for $\rho = \frac{n-k}{n}$.

3 AMD codes for leaky storage

We consider codes over a finite field \mathbb{F}_q , where q is a prime power, and assume message set $\mathcal{M} = \mathbb{F}_q^k$ and the storage stores an element of the group $\mathcal{G} = \mathbb{F}_q^n$.

3.1 Definition of ρ -AMD

Definition 5. An (n, k) -coding scheme consists of two functions: a randomized encoding function $Enc : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^n$, and deterministic decoding function $Dec : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^k \cup \{\perp\}$, satisfying $\Pr[Dec(Enc(\mathbf{m})) = \mathbf{m}] = 1$, for any $\mathbf{m} \in \mathbb{F}_q^k$. Here probability is taken over the randomness of the encoding algorithm.

The information rate of an (n, k) -coding scheme is $\frac{k}{n}$.

We now define our leakage model and codes that detect manipulation in presence of this leakage. Let $\mathbf{X} = Enc(\mathbf{m})$ for a message $\mathbf{m} \in \mathcal{M}$, and $\mathbb{A}_{\mathbf{Z}}$ denote an adversary with access to a variable \mathbf{Z} , representing the leakage of information about the codeword.

Definition 6 (ρ -AMD). An (n, k) -coding scheme is called a strong ρ -AMD code with security parameter δ if $\Pr[Dec(\mathbb{A}_{\mathbf{Z}}(Enc(\mathbf{m}))) \notin \{\mathbf{m}, \perp\}] \leq \delta$ for any message $\mathbf{m} \in \mathbb{F}_q^k$ and adversary $\mathbb{A}_{\mathbf{Z}}$ whose leakage variable \mathbf{Z} satisfies $\tilde{H}_{\infty}(\mathbf{X}|\mathbf{Z}) \geq H_{\infty}(\mathbf{X}) - \rho n \log q$, and is allowed to choose any offset vector in \mathbb{F}_q^n to add to the codeword.

The code is called a weak ρ -AMD code if security holds for $\mathbf{M} \xleftarrow{\$} \mathbb{F}_q^k$ (rather than an arbitrary message distribution). The encoder in this case is deterministic and the probability of outputting a different message is over the randomness of the message.

A family $\{(Enc_n, Dec_n)\}$ of ρ -AMD codes is a set of $(n, k(n))$ -coding schemes indexed by the codeword length n , where for any value of δ , there is an $N \in \mathbb{N}$ such that for all $n \geq N$, (Enc_n, Dec_n) is a ρ -AMD code with security parameter δ .

A rate R is achievable if there exists a family $\{(Enc_n, Dec_n)\}$ of ρ -AMD codes such that $\lim_{n \rightarrow \infty} \frac{k(n)}{n} = R$ as δ approaches 0

Our definition bounds the amount of leakage in comparison with an adversary who observes up to ρn components of the stored codeword. We call this latter adversary a *Limited-View (LV) adversary* [3]. According to Lemma 1, the min-entropy of the stored codeword given an LV-adversary will be $\tilde{H}_{\infty}(\mathbf{X}|\mathbf{Z}) \geq H_{\infty}(\mathbf{X}) - \rho n \log q$. We require the same min-entropy be left in the codeword, for an arbitrary leakage variable \mathbf{Z} accessible to the adversary.

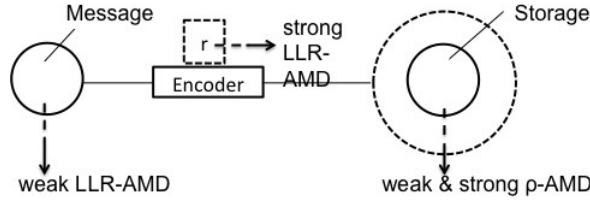


Fig. 1. The arrow shows the part of the system that leaks.

Fig. 1 shows places of leakage in AMD encoding in our model, and the models in Definition 2 and Definition 3.

Proposition 1. *Let \mathbf{X} denote a random variable representing the codeword of a message \mathbf{m} (\mathbf{M} for weak codes), and \mathbf{Z} denote the leakage variable of the adversary $\mathbb{A}_{\mathbf{Z}}$ who uses the leakage information to construct the best offset vector to make the decoder output a different message. For a ρ -AMD code with security parameter δ , we have $\tilde{H}(\mathbf{X}|\mathbf{Z}) \geq \log \frac{1}{\delta}$.*

Proof. We write the proof for strong ρ -AMD codes. (The proof for weak ρ -AMD codes follows similarly.) According to the security definition of ρ -AMD codes, we have

$$\Pr[\text{Dec}(\mathbb{A}_{\mathbf{Z}}(\mathbf{X})) \notin \{\mathbf{m}, \perp\}] \leq \delta,$$

where the probability is over the randomness of \mathbf{X} , and is the expectation over $\mathbf{z} \in \mathcal{Z}$. If the adversary with the leakage variable $\mathbf{Z} = \mathbf{z}$ can correctly guess the value \mathbf{x} of \mathbf{X} , then a codeword \mathbf{x}' corresponding to another message \mathbf{m}' can be constructed to cause the decoder to output \mathbf{m}' , by using $\mathbb{A}_{\mathbf{z}}(\mathbf{X}) = \mathbf{X} + (\mathbf{x}' - \mathbf{x})$. We then have

$$\Pr[\text{Dec}(\mathbb{A}_{\mathbf{Z}}(\mathbf{X})) \notin \{\mathbf{m}, \perp\} | \mathbf{Z} = \mathbf{z}] \geq \max_{\mathbf{x}} \Pr[\mathbf{X} = \mathbf{x} | \mathbf{Z} = \mathbf{z}],$$

which by taking expectation over $\mathbf{z} \in \mathcal{Z}$ yields

$$\mathbb{E}_{\mathbf{z}} (\Pr[\text{Dec}(\mathbb{A}_{\mathbf{Z}}(\mathbf{X})) \notin \{\mathbf{m}, \perp\} | \mathbf{Z} = \mathbf{z}]) \geq \mathbb{E}_{\mathbf{z}} \left(\max_{\mathbf{x}} \Pr[\mathbf{X} = \mathbf{x} | \mathbf{Z} = \mathbf{z}] \right) = 2^{-\tilde{H}(\mathbf{X}|\mathbf{Z})},$$

The last equality follows from the definition of conditional min-entropy. The desired inequality then follows directly from the security definition of ρ -AMD codes as follows.

$$2^{-\tilde{H}(\mathbf{X}|\mathbf{Z})} \leq \Pr[\text{Dec}(\mathbb{A}_{\mathbf{Z}}(\mathbf{X})) \notin \{\mathbf{m}, \perp\} | \mathbf{Z} = \mathbf{z}] \leq \delta \iff \tilde{H}(\mathbf{X}|\mathbf{Z}) \geq \log \frac{1}{\delta}.$$

□

Definition 7. *Let \mathcal{C} denote the set of codewords of a code, and $\mathcal{C}_{\mathbf{m}}$ denote the set of codewords corresponding to the message \mathbf{m} , i.e. $\mathcal{C}_{\mathbf{m}} = \{\text{Enc}(\mathbf{m}, \mathbf{r}) | \mathbf{r} \in \mathcal{R}\}$. A randomised encoder is called regular if $|\mathcal{C}_{\mathbf{m}}| = |\mathcal{R}|$ for all \mathbf{m} .*

We note that because the code has zero decoding error when there is no adversary corruption, we have

$$\mathcal{C}_{\mathbf{m}} \cap \mathcal{C}_{\mathbf{m}'} = \emptyset, \forall \mathbf{m}, \mathbf{m}' \in \mathcal{M}. \quad (4)$$

This means that for regular randomised encoders, a codeword uniquely determines a pair (\mathbf{m}, \mathbf{r}) . Assuming that the randomized encoder uses r uniformly distributed bits, the random variable $\mathbf{X} = \text{Enc}(\mathbf{m}, \mathbf{R})$ is flat over $\mathcal{C}_{\mathbf{m}}$.

Lemma 4. *The relations between Strong LLR-AMD codes and strong ρ -AMD codes are as follows.*

1. If there exists a regular randomized encoder for a $(q^k, q^n, 2^r, \alpha, \delta)$ -strong LLR-AMD code, then there is an encoder for strong ρ -AMD code with security parameter δ and leakage parameter ρ where $\rho \leq \frac{\alpha r}{n \log q}$.
2. If there exists a regular randomized encoder for a strong ρ -AMD code with security parameter δ and leakage parameter ρ , then there is an encoder for a $(q^k, q^n, 2^r, \alpha, \delta)$ -strong LLR-AMD code with α and r where $\alpha \leq \frac{n \rho \log q}{r}$ and $r \geq \log \frac{1}{\delta} + n \rho \log q$.

Proof of Lemma 4 is given in Appendix A.

In [2], it is shown that the optimal AMD code in Construction 1 gives a $(q^d, q^{d+2}, q, \alpha, \frac{d+1}{q^{1-\alpha}})$ -strong LLR-AMD code. The parameters of this LLR-AMD code are $k = d$, $n = d + 2$, $r = \log q$ and $\delta = \frac{d+1}{q^{1-\alpha}}$. A simple mathematical manipulation of these equations gives $\alpha = 1 - \log_q \frac{n-1}{\delta}$, and substituting them into Lemma 4, item 1, we obtain

$$\rho \leq \frac{(1 - \log_q \frac{n-1}{\delta}) \log q}{n \log q} = \frac{1 - \log_q \frac{n-1}{\delta}}{n}.$$

This results in the following.

Corollary 1. *The code in Construction 1 is a strong ρ -AMD code with $k = d$, $n = d + 2$, security parameter δ and leakage parameter $\rho \leq \frac{1 - \log_q \frac{n-1}{\delta}}{n}$.*

It is easy to see that $\rho < \frac{1}{n}$. Thus the resulting construction of strong ρ -AMD codes can only tolerate a very small leakage. Moreover the upper bound on ρ vanishes as n goes to infinity and so this construction cannot give a non-trivial family of strong ρ -AMD code. We note that the same construction resulted in a family of strong LLR-AMD codes with asymptotic rate 1.

Lemma 5. *The relations between weak LLR-AMD codes and weak ρ -AMD codes are as follows.*

1. A $(q^k, q^n, \alpha, \delta)$ -weak LLR-AMD code is a weak ρ -AMD code with security parameter δ and leakage parameter ρ satisfying $\rho \leq \frac{\alpha k}{n}$.
2. A weak ρ -AMD code with security parameter δ and leakage parameter ρ is a $(q^k, q^n, \alpha, \delta)$ -weak LLR-AMD code satisfying $\alpha \leq \frac{\rho n}{k}$.

Proof of Lemma 5 is given in Appendix B.

A construction of $(q^d, q^{d+1}, \alpha, \frac{2}{q^{1-\alpha d}})$ weak LLR-AMD codes is given in [2, Theorem 2]. The code has parameters $k = d$, $n = d + 1$ and $\delta = \frac{2}{q^{1-\alpha d}}$. A simple mathematical manipulation of these equations gives $\alpha = \frac{1 - \log_q \frac{2}{\delta}}{n-1}$, and so from Lemma 5, item 1, we obtain

$$\rho \leq \frac{(\frac{1 - \log_q \frac{2}{\delta}}{n-1})(n-1)}{n} = \frac{1 - \log_q \frac{2}{\delta}}{n}.$$

Corollary 2. *The code in [2, Theorem 2] is a weak ρ -AMD code with $k = d$, $n = d + 1$, security parameter δ and leakage parameter $\rho \leq \frac{1 - \log_q \frac{2}{\delta}}{n}$.*

This construction gives ρ -AMD codes with small ρ , and cannot be used to construct a family of ρ -AMD codes for $\rho > 0$.

3.2 Efficiency bounds for ρ -AMD codes

Theorem 1. *If an (n, k) -coding scheme is a strong ρ -AMD code with security parameter δ , then,*

$$k \leq n(1 - \rho) + \frac{2 \log \delta - 1}{\log q}. \quad (5)$$

The achievable rate of strong ρ -AMD codes is upper bounded by $1 - \rho$.

Proof. Consider a strong ρ -AMD code with security parameter δ . By Proposition 1, $\tilde{H}_\infty(\mathbf{X}|\mathbf{Z}) \geq \log \frac{1}{\delta}$ should hold for any \mathbf{Z} satisfying $\tilde{H}_\infty(\mathbf{X}|\mathbf{Z}) \geq H_\infty(\mathbf{X}) - \rho n \log q$. In particular, the inequality should hold for \mathbf{Z} such that $\tilde{H}_\infty(\mathbf{X}|\mathbf{Z}) = H_\infty(\mathbf{X}) - \rho n \log q$. We then have $H_\infty(\mathbf{X}) - \rho n \log q \geq \log \frac{1}{\delta}$. On the other hand, we always have $\log |\mathcal{C}_{\mathbf{m}}| \geq H_\infty(\mathbf{X})$, where $\mathcal{C}_{\mathbf{m}}$ denotes the set of codewords corresponding to message \mathbf{m} , which is the support of \mathbf{X} . This gives the following lower bound on $|\mathcal{C}_{\mathbf{m}}|$.

$$|\mathcal{C}_{\mathbf{m}}| \geq \frac{2^{\rho n \log q}}{\delta} = \frac{q^{\rho n}}{\delta}. \quad (6)$$

Now consider the adversary randomly choose an offset $\Delta \neq 0^n$, we have

$$\begin{aligned} \delta &\geq \Pr[\text{Enc}(\mathbf{m}) + \Delta \in \cup_{\mathbf{m}' \neq \mathbf{m}} E_{\mathbf{m}'}] \\ &\geq \frac{|\cup_{\mathbf{m}' \neq \mathbf{m}} E_{\mathbf{m}'}|}{|\mathbb{F}_q^n| - 1} \\ &\stackrel{(4),(6)}{\geq} \frac{(q^k - 1) \cdot \frac{q^{\rho n}}{\delta}}{q^n - 1}. \end{aligned} \quad (7)$$

Therefore,

$$k \leq n(1 - \rho) + \frac{2 \log \delta - 1}{\log q}.$$

□

Proposition 2. *If an (n, k) -coding scheme is a weak ρ -AMD code with security parameter δ , then $q^{\rho n - k} \leq \delta$ and $\frac{q^k - 1}{q^n - 1} \leq \delta$.*

Proof of Proposition 2 is given in Appendix C.

4 Limited-View ρ -AMD codes

We consider a special type of leakage where the adversary chooses a subset $S, |S| = \rho n$ (n is the codeword length), and the codeword components associated with this set will be revealed to them. The adversary will then use this information to construct their offset vector. A tampering strategy is a function from \mathbb{F}_q^n to \mathbb{F}_q^n which can be described by the notation $f_{S,g}$, where $S \subset [n]$ and a function $g : \mathbb{F}_q^{n\rho} \rightarrow \mathbb{F}_q^n$, with the following interpretation. The set S specifies a subset of ρn indexes of the codeword that the adversary choose. The function g determines an offset for each read value on the subset S . A ρ^{LV} -AMD code provides protection against all adversary strategies. (This approach to defining tampering functions is inspired by Non-Malleable Codes (NMC) [11].)

Let $\mathcal{S}^{[n\rho]}$ be the set of all subsets of $[n]$ of size $n\rho$. Let $\mathcal{M}(\mathbb{F}_q^{n\rho}, \mathbb{F}_q^n)$ denote the set of all functions from $\mathbb{F}_q^{n\rho}$ to \mathbb{F}_q^n , namely, $\mathcal{M}(\mathbb{F}_q^{n\rho}, \mathbb{F}_q^n) := \{g : \mathbb{F}_q^{n\rho} \rightarrow \mathbb{F}_q^n\}$.

Definition 8 (\mathcal{F}_ρ^{add}). *The class of tampering function \mathcal{F}_ρ^{add} , consists of the set of functions $\mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$, that can be described by two parameters, $S \in \mathcal{S}^{[n\rho]}$ and $g \in \mathcal{M}(\mathbb{F}_q^{n\rho}, \mathbb{F}_q^n)$. The set \mathcal{F}_ρ^{add} of limited view algebraic tampering functions are defined as follows.*

$$\mathcal{F}_\rho^{add} = \left\{ f_{S,g}(\mathbf{x}) \mid S \in \mathcal{S}^{[n\rho]}, g \in \mathcal{M}(\mathbb{F}_q^{n\rho}, \mathbb{F}_q^n) \right\}, \quad (8)$$

where $f_{S,g}(\mathbf{x}) = \mathbf{x} + g(\mathbf{x}_{|S})$ for $\mathbf{x} \in \mathbb{F}_q^n$.

Definition 9 (ρ^{LV} -AMD). *An (n, k) -coding scheme is called a strong ρ^{LV} -AMD code with security parameter δ if $\Pr[\text{Dec}(f(\text{Enc}(\mathbf{m}))) \notin \{\mathbf{m}, \perp\}] \leq \delta$ for any message $\mathbf{m} \in \mathbb{F}_q^k$ and any $f_{S,g} \in \mathcal{F}_\rho^{add}$. It is called a weak ρ^{LV} -AMD code if it only requires the security to hold for a random message $\mathbf{M} \leftarrow \mathbb{F}_q^k$ rather than an arbitrary message \mathbf{m} .*

We first give a generic construction of strong ρ^{LV} -AMD codes from WtII codes and AMD codes.

Construction 3 *Let $(\text{AMDenc}, \text{AMDdec})$ be a $(q^k, q^{n'}, \delta)$ -AMD code and let $(\text{WtIIenc}, \text{WtIIdec})$ be a linear $(\rho, 0)$ -wiretap II code with encoder $\text{WtIIenc} : \mathbb{F}_q^{n'} \rightarrow \mathbb{F}_q^n$. Then (Enc, Dec) defined as follows is a strong ρ^{LV} -AMD codes with security parameter δ .*

$$\begin{cases} \text{Enc}(\mathbf{m}) = \text{WtIIenc}(\text{AMDenc}(\mathbf{m})); \\ \text{Dec}(\mathbf{x}) = \text{AMDdec}(\text{WtIIdec}(\mathbf{x})). \end{cases}$$

When instantiated with the $(q^k, q^{k+2}, \frac{k+1}{q})$ -AMD code in Construction 1 and the linear $(\rho, 0)$ -wiretap II code in Construction 2, we obtain a family of strong ρ^{LV} -AMD codes with security parameter $\frac{k+1}{q}$ that achieves rate $1 - \rho$.

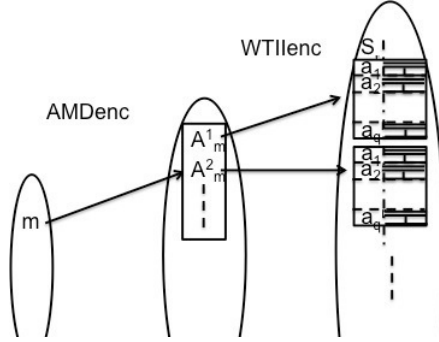


Fig. 2. WtIIoAMD construction with $A_{\mathbf{m}}^i$ denoting the values of $\text{AMDenc}(\mathbf{m})$

Proof. Since both AMDenc and WtIIenc are randomised encoders, in this proof we write the randomness of a randomized encoder explicitly. Let I denote the randomness of AMDenc and let J denote the randomness of WtIIenc . As illustrated in Fig. 2, a message \mathbf{m} is first encoded into an AMD codeword $A_{\mathbf{m}}^I = \text{AMDenc}(\mathbf{m}, I)$. The AMD codeword $A_{\mathbf{m}}^I$ is then further encoded into a WtII codeword, which is the final ρ^{LV} -AMD codeword: $\text{Enc}(\mathbf{m}) = \text{WtIIenc}(A_{\mathbf{m}}^I, J)$. According to (3), $\text{SD}(\text{WtIIenc}(A_{\mathbf{m}}^{i_1}, J)_{|S}; \text{WtIIenc}(A_{\mathbf{m}}^{i_2}, J)_{|S}) = 0$. This says that $A_{\mathbf{m}}^I$ and $\text{Enc}(\mathbf{m})_{|S}$ are independent random variables, in particular, \mathbf{I} and $\text{Enc}(\mathbf{m})_{|S}$ are independent. According to Definition 9, to show that (Enc, Dec) is a strong ρ^{LV} -AMD code with security parameter δ , we need to show that for any message \mathbf{m} , and any $f_{S,g} \in \mathcal{F}_{\rho}^{add}$, $\Pr[\text{Dec}(f_{S,g}(\text{Enc}(\mathbf{m}))) \notin \{\mathbf{m}, \perp\}] \leq \delta$, where the probability is over the randomness (\mathbf{I}, \mathbf{J}) of the encoder Enc . We show this in two steps.

Step 1. In this step, we assume that $\text{Enc}(\mathbf{m})_{|S} = \mathbf{a}$ has occurred and bound the error probability of (Enc, Dec) under this condition. We compute

$$\begin{aligned}
 & \Pr[\text{Dec}(f_{S,g}(\text{Enc}(\mathbf{m}))) \notin \{\mathbf{m}, \perp\} | (\text{Enc}(\mathbf{m})_{|S} = \mathbf{a})] \\
 &= \Pr[\text{Dec}(\text{Enc}(\mathbf{m}) + g(\mathbf{a})) \notin \{\mathbf{m}, \perp\} | (\text{Enc}(\mathbf{m})_{|S} = \mathbf{a})] \\
 &= \Pr[\text{AMDdec}(\text{WtIIdec}(\text{WtIIenc}(\text{AMDenc}(\mathbf{m}, \mathbf{I}), \mathbf{J}) + g(\mathbf{a}))) \notin \{\mathbf{m}, \perp\} \\
 &\quad | (\text{Enc}(\mathbf{m})_{|S} = \mathbf{a})] \\
 &= \Pr[\text{AMDdec}(\text{AMDenc}(\mathbf{m}, \mathbf{I}) + \text{WtIIdec}(g(\mathbf{a}))) \notin \{\mathbf{m}, \perp\} | (\text{Enc}(\mathbf{m})_{|S} = \mathbf{a})] \\
 &= \Pr[\text{AMDdec}(\text{AMDenc}(\mathbf{m}, \mathbf{I}) + \text{WtIIdec}(g(\mathbf{a}))) \notin \{\mathbf{m}, \perp\}], \\
 &\leq \delta,
 \end{aligned}$$

where the third equality follows from the linearity of $(\text{WtIIenc}, \text{WtIIdec})$, the last equality follows from the fact that \mathbf{I} and $\text{Enc}(\mathbf{m})_{|S}$ are independent discussed in the beginning of the proof, and the inequality follows trivially from the security of $(\text{AMDenc}, \text{AMDdec})$.

Step 2. In this step, we conclude the first part of the proof by showing

$$\begin{aligned}
& \Pr[\text{Dec}(f_{S,g}(\text{Enc}(\mathbf{m}))) \notin \{\mathbf{m}, \perp\}] \\
&= \sum_{\mathbf{a}} \Pr[\text{Enc}(\mathbf{m})|_S = \mathbf{a}] \cdot \Pr[\text{Dec}(f_{S,g}(\text{Enc}(\mathbf{m}))) \notin \{\mathbf{m}, \perp\} | (\text{Enc}(\mathbf{m})|_S = \mathbf{a})] \\
&\leq \sum_{\mathbf{a}} \Pr[\text{Enc}(\mathbf{m})|_S = \mathbf{a}] \cdot \delta \\
&= \delta,
\end{aligned}$$

where the inequality follows from **Step 1**.

Finally, the rate of the $(\rho, 0)$ -wiretap II code in Construction 2 is $\frac{k+2}{n} = 1 - \rho$. So the asymptotic rate of the strong ρ^{LV} -AMD code family is

$$\lim_{n \rightarrow \infty} \frac{k}{n} = \lim_{n \rightarrow \infty} \frac{(1 - \rho)n - 2}{n} = 1 - \rho.$$

□

We next show a construction of weak ρ^{LV} -AMD codes that achieves asymptotic rate 1.

Construction 4 Let \mathbb{F}_q be a finite field of q elements. Let G be a $k \times k$ non-singular matrix over \mathbb{Z}_{q-1} such that each column of G consists of distinct entries, i.e., $g_{i,j} \neq g_{i',j}$ for any j and $i \neq i'$. Assume the entries of G (viewed as integers) is upper-bounded by ψk for constant ψ , i.e., $g_{i,j} \leq \psi k$. Then the following construction gives a family of weak LV-AMD codes of asymptotic rate 1 with any leakage parameter $\rho < 1$.

$$\text{Enc}^1 : (\mathbb{F}_q^*)^k \rightarrow (\mathbb{F}_q^*)^k \times \mathbb{F}_q : \mathbf{m} \mapsto (\mathbf{m} || f(\mathbf{m}, G)),$$

where \mathbb{F}_q^* denotes the set of non-zero elements of \mathbb{F}_q and the tag $f(\mathbf{m}, G)$ is generated as follows.

$$f(\mathbf{m}, G) = \sum_{j=1}^k \prod_{i=1}^k m_i^{g_{i,j}}. \quad (9)$$

The decoder dec checks if the first k -tuple of the input vector, when used in 9, match the last component.

The proof of Construction 4 is given in Appendix D.

Concrete constructions of the matrix G can be found in [2, Remark 2].

5 Applications

5.1 Robust Ramp SSS

A *Secret Sharing Scheme (SSS)* consists of two algorithms (Share, Recover). The algorithm Share maps a secret $\mathbf{s} \in \mathcal{S}$ to a vector $\mathbf{S} = (S_1, \dots, S_N)$ where the

¹ The message distribution in this construction is not exactly uniform over \mathbb{F}_q^k but $(\mathbb{F}_q^*)^k$. So this construction can achieve security even when the message distribution is not uniform.

shares S_i are in some set \mathcal{S}_i and will be given to participant P_i . The algorithm Recover takes as input a vector of shares $\tilde{\mathbf{S}} = (\tilde{S}_1, \dots, \tilde{S}_N)$ where $\tilde{S}_i \in \mathcal{S}_i \cup \{\perp\}$, where \perp denotes an absent share. For a (t, N) -threshold SSS, t shares reveal no information about the secret \mathbf{s} and $t + 1$ shares uniquely recover the secret \mathbf{s} . For a (t, r, N) -ramp SSS [7] with $(\text{Share}_{\text{rsss}}, \text{Recover}_{\text{rsss}})$ as sharing and recovering algorithms, the access structure is specified by two thresholds. The privacy threshold is t , and the reconstruction threshold is r . In a (t, r, N) -ramp SSS, subsets of t or less shares do not reveal any information about the secret, and subsets of r or more shares can uniquely recover the secret \mathbf{s} . A set of shares of size $t < a < r$ may leak some information about the secret. In particular, we consider ramp schemes in which a set of $t + \alpha(r - t)$ shares leak α fraction of secret information.

Definition 10 ((t, r, N)-Ramp Secret Sharing Scheme). A (t, r, N) -ramp secret sharing scheme is consist of a pair of algorithms $(\text{Share}_{\text{rsss}}, \text{Recover}_{\text{rsss}})$, where $\text{Share}_{\text{rsss}}$ randomly maps a secret $\mathbf{s} \in \mathcal{S}$ to a share vector $\mathbf{S} = (S_1, \dots, S_N)$ and $\text{Recover}_{\text{rsss}}$ deterministically reconstruct a $\tilde{\mathbf{s}} \in \mathcal{S}$ or output \perp , satisfy the following.

- *Privacy:* The adversary can access up to $r - 1$ shares. If the number of shares accessed by the adversary is $a \leq t$, no information will be leaked about the secret. If the number of leaked share is $a = t + \alpha(r - t)$, where $0 < \alpha < 1$, then $H_\infty(S|S_{i_1} \dots S_{i_a}) \geq H_\infty(S) - \alpha \log |\mathcal{S}|^2$, for $S \leftarrow \mathcal{S}$ and any $\{i_1, \dots, i_a\} \subset [N]$.
- *Reconstruction:* Any r correct shares can reconstruct the secret \mathbf{s} .

A linear ramp SSS has the additional property that the Recover function is linear, namely, for any $\mathbf{s} \in \mathcal{G}$, any share vector \mathbf{S} of \mathbf{s} , and any vector \mathbf{S}' (possibly containing some \perp symbols), we have $\text{Recover}_{\text{rsss}}(\mathbf{S} + \mathbf{S}') = \mathbf{s} + \text{Recover}_{\text{rsss}}(\mathbf{S}')$, where vector addition is defined element-wise and addition with \perp is defined by $\perp + \mathbf{x} = \mathbf{x} + \perp = \perp$ for all \mathbf{x} . In a linear SSS, the adversary can modify the shares $\tilde{S}_i = S_i + \Delta_i$, such that the difference $\Delta = \tilde{\mathbf{s}} - \mathbf{s}$ between the reconstructed secret and the shared secret, is known.

In a (t, N, δ) -robust SSS, for any $t + 1$ shares with at most t shares modified by the adversary, the reconstruction algorithm can recover the secret \mathbf{s} , or detect the adversarial modification and output \perp , with probability at least $1 - \delta$ [1]. That is with probability at most δ the secret is either not recoverable, or a wrong secret is accepted. A modular construction of the robust SSS using an AMD code and a linear SSS is given by Cramer *et al.* [1].

We define robust ramp secret sharing scheme when the adversary can adaptively corrupt up to $t + \rho(r - t)$ shares, where $0 < \rho < 1$ is a constant (level of robustness against active adversaries).

² This definition of leakage is seemingly different from [13], where uniform distribution of secret S is assumed and Shannon entropy is used instead of min-entropy.

Definition 11 $((t, r, N, \rho, \delta)$ -Robust Ramp Secret Sharing Scheme). A $((t, r, N, \rho, \delta)$ -robust ramp secret sharing scheme is consist of a pair of algorithms $(\text{Share}_{\text{rrss}}, \text{Recover}_{\text{rrss}})$, where $\text{Share}_{\text{rrss}}$ randomly maps a secret $\mathbf{s} \in \mathcal{S}$ to a share vector $\mathbf{S} = (S_1, \dots, S_N)$ and $\text{Recover}_{\text{rrss}}$ deterministically reconstruct a $\tilde{\mathbf{s}} \in \mathcal{S}$ or output \perp , satisfy the following.

- Privacy: The adversary can access up to $r - 1$ shares. If the number of shares accessed by the adversary is $a \leq t$, no information will be leaked about the secret. If the number of leaked share is $a = t + \alpha(r - t)$, where $0 < \alpha < 1$, then $\tilde{H}_\infty(S|S_{i_1} \dots S_{i_a}) \geq H_\infty(S) - \alpha \log |\mathcal{S}|$, for $S \leftarrow \mathcal{S}$ and any $\{i_1, \dots, i_a\} \subset [N]$.
- Reconstruction: Any r correct shares can reconstruct the secret s .
- Robustness: For any r shares with at most $t + \rho(r - t)$ corrupted shares, the probability that either the secret is correctly reconstructed, or the the adversary's modifications being detected, is at least $1 - \delta$.

We propose a general construction of robust ramp secret sharing scheme using a ρ_{amd} -AMD and (t, r, N) -ramp secret sharing scheme.

Theorem 2. Consider a linear (t, r, N) -ramp secret sharing scheme with the algorithm pair $(\text{Share}_{\text{rssh}}, \text{Recover}_{\text{rssh}})$ and shares $\mathcal{S}_i \in \mathbb{F}_q^m$, $i = 1, \dots, N$, and let (Enc, Dec) be a ρ_{amd} -AMD code $\mathbb{F}_q^k \rightarrow \mathbb{F}_q^n$, with failure probability δ_{amd} and $n = (r - t)m$. Then there is a robust ramp secret sharing scheme with algorithm pair $(\text{Share}_{\text{rrss}}, \text{Recover}_{\text{rrss}})$ given by $\text{Share}_{\text{rrss}}(\mathbf{s}) = \text{Share}_{\text{rssh}}(\text{Enc}(\mathbf{s}))$ and $\text{Recover}_{\text{rrss}}(\tilde{\mathbf{S}}) = \text{Dec}(\text{Recover}_{\text{rssh}}(\tilde{\mathbf{S}}))$ is a $((t, r, N, \rho, \delta)$ -Robust Ramp Secret Sharing Scheme with $\rho \leq \rho_{\text{amd}}$ and $\delta \leq \delta_{\text{amd}}$.

Proof. First, we show that if the adversary reads at most $t + \rho(r - t)$ shares, the ρ_{amd} -AMD codeword c leaks at most $\rho n \log q$ informations. Since the ρ_{amd} -AMD codeword is encoded by a (t, r, N) ramp secret sharing scheme, t shares will not leak any information about the ρ_{amd} -AMD codeword c . Given that the share size $|\mathcal{S}_i| \leq q^m$ and $n = (r - t)m$, the leakage of the extra $\rho(r - t)$ shares will leak at most $\rho n \log q$ bit of information about the ρ_{amd} -AMD codeword c .

Second, we show that the resulting secret sharing scheme is δ -robust. For a secret \mathbf{s} , let $\mathbf{S} \leftarrow \text{Share}_{\text{rrss}}(\mathbf{s})$ be the original share vector and $\tilde{\mathbf{S}}$ be the corrupted one, and let $\mathbf{S}' = \tilde{\mathbf{S}} - \mathbf{S}$. For any r shares, the failure probability of the reconstruction is given by,

$$\Pr[\text{Recover}_{\text{rrss}}(\tilde{\mathbf{S}}) \notin \{\mathbf{s}, \perp\}] \stackrel{(1)}{=} \Pr[\text{Dec}(\text{Recover}_{\text{rssh}}(\mathbf{S}) + \text{Recover}_{\text{rssh}}(\mathbf{S}')) \notin \{\mathbf{s}, \perp\}] \\ = \Pr[\text{Dec}(\text{Enc}(\mathbf{s}) + \Delta) \notin \{\mathbf{s}, \perp\}],$$

where $\Delta = \text{Recover}_{\text{rssh}}(\mathbf{S}')$ is chosen by the adversary \mathbb{A} , and (1) uses the linearity of the ramp scheme. In choosing Δ , the adversary \mathbb{A} can use at most ρ fraction of information in the ρ_{amd} -AMD codeword $c = \text{Enc}(\mathbf{s})$. Since at most $\rho n \log q$ information bit is leaked to the adversary, that is $\tilde{H}_\infty(C|\mathbf{Z}) \geq H_\infty(C) - \rho n \log q$, from the definition of ρ_{amd} -AMD code with $\rho \leq \rho_{\text{amd}}$, the decoding algorithm Dec outputs correct secret \mathbf{s} , or detects the error \perp , with probability at least

$1 - \delta_{\text{amd}}$. This means that the ramp secret sharing scheme is robust and outputs either the correct secret, or detects the adversarial tampering, with probability at most $1 - \delta \geq 1 - \delta_{\text{amd}}$. Thus a (t, r, N) -ramp secret sharing scheme and a ρ_{amd} -AMD with security parameter δ_{amd} , give a (t, r, N, ρ, δ) -robust ramp secret sharing scheme with $\delta \leq \delta_{\text{amd}}$ and $\rho \leq \rho_{\text{amd}}$. \square

5.2 Wiretap II with Algebraic Adversary

The Wiretap II [5] problem considers a passive adversary that can read a ρ fraction of the codeword components and the goal is to prevent the adversary from learning information about the sent message. Wiretap II with an active adversary has been considered in [14] and later generalized in [15,8]. In this latter general model, called Adversarial Wiretap (AWTP) mode, the adversary is characterized by two parameters ρ_r and ρ_w , denoting the fraction of the codeword components the adversary can “read” and “modify additively”, respectively. The goal is two-fold: to prevent the adversary from obtaining any information (secrecy) and, to recover the message despite the changes made by the adversary (reliability). It was proved [8] that in AWTP model, where the adversary can write to a ρ_w fraction of the codeword components additively, secure and reliable communication is possible if, $\rho_r + \rho_w < 1$. This says that when $\rho_r + \rho_w > 1$, one can only hope for weaker type of security, for example, secrecy and error detection. We consider wiretap II with an algebraic adversary, who can read a ρ fraction of the codeword components and tamper with the whole codeword algebraically, namely, adding a non-zero group element to the codeword (codewords are assumed to live in a group). The adversary in this model is equivalent to the AWTP adversary with $\rho_r = \rho$ and $\rho_w = 1$. But the coding goal of wiretap II with an algebraic adversary is different from AWTP.

Definition 12. *An algebraic tampering wiretap II channel is a communication channel between Alice and Bob that is (partially) controlled by an adversary Eve with two following capabilities.*

- *Read:* Eve adaptively selects a fraction ρ of the components of the transmitted codeword $\mathbf{c} = c_1, \dots, c_n$ to read, namely, Eve’s knowledge of the transmitted codeword is given by $\mathbf{Z} = \{c_{i_1}, \dots, c_{i_{\rho n}}\}$, where $S = \{i_1, \dots, i_{\rho n}\} \subset [n]$ is chosen by Eve.
- *Write:* Assume $\mathbf{c} \in \mathcal{G}$ for some additive group \mathcal{G} . Eve chooses an “off-set” $\Delta \in \mathcal{G}$ according to \mathbf{Z} and add it to the codeword \mathbf{c} , namely, the channel outputs $\mathbf{c} + \Delta$.

Definition 13 $((\rho, \epsilon, \delta)$ -algebraic tampering wiretap II $((\rho, \epsilon, \delta)$ -AWtII)). *A (ρ, ϵ, δ) -AWtII code is a coding scheme (Enc, Dec) that guarantees the following two properties.*

- *Secrecy:* For any pair of messages \mathbf{m}_0 and \mathbf{m}_1 , any $S \subset [n]$ of size $|S| \leq n\rho$, (3) should hold, namely,

$$\text{SD}(\text{Enc}(\mathbf{m}_0)|_S; \text{Enc}(\mathbf{m}_1)|_S) \leq \epsilon.$$

- Robustness: *If the adversary is passive, Dec always outputs the correct message. If the adversary is active, the probability that the decoder outputs a wrong message is bounded by δ . That is, for any message \mathbf{m} and any ρ -algebraic tampering wiretap II adversary \mathbb{A} ,*

$$\Pr[\text{Dec}(\mathbb{A}(\text{Enc}(\mathbf{m}))) \notin \{\mathbf{m}, \perp\}] \leq \delta.$$

The secrecy of (ρ, ϵ, δ) -AWtII code implies that a (ρ, ϵ, δ) -AWtII code is a (ρ, ϵ) -WtII code. The following rate upper bound follows directly from Lemma 3.

Corollary 3. *The rate of $(\rho, 0, \delta)$ -AWtII codes is bounded by $R \leq 1 - \rho$.*

The robustness property of (ρ, ϵ, δ) -AWtII code is the same as the security of a strong ρ^{LV} -AMD code (see Definition 9). Furthermore, the construction of ρ^{LV} -AMD codes in Construction 3 uses a $(\rho, 0)$ -WtII code to encode $\mathbf{c} = \text{AMDenc}(\mathbf{m})$, which guarantees secrecy with respect to any pair of $(\mathbf{c}_0, \mathbf{c}_1)$, and hence secrecy with respect to any pair of $(\mathbf{m}_0, \mathbf{m}_1)$. These assert that Construction 3 yields a family of $(\rho, 0, \delta)$ -AWtII codes.

Corollary 4. *There exists a family of $(\rho, 0, \delta)$ -AWtII codes that achieves rate $R = 1 - \rho$.*

6 Conclusion

We considered an extension of AMD codes when the storage leaks information and the amount of leaked information is bounded by $\rho \log |\mathcal{G}|$. We defined ρ -AMD codes that provide protection in this scenario, both with weak and strong security, and derived concrete and asymptotic bounds on the efficiency of codes in these settings. Table 1 compares our results with original AMD codes and an earlier work (called LLR-AMD) that allow leakage in specific parts of the encoding process. Unlike LLR-AMD that uses different leakage requirements for the weak and strong case, we use a single model to express the leakage and require that the left-over entropy of the codeword be lower bounded. This makes our analysis and constructions more challenging. In particular, optimal constructions of LLR-AMD codes follow directly from the optimal constructions of original AMD codes. However constructing optimal ρ -AMD code, in both weak and strong model, remain open. We gave an explicit construction of a family of codes with respect to a weaker notion of leakage (ρ^{LV} -AMD) whose rate achieves the upper bounds of the ρ -AMD codes. We finally gave two applications of the codes to robust ramp secret sharing schemes and algebraic manipulation wiretap II channel.

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Appendices

A Proof of Lemma 4

Proof. Assume a regular encoder and consider a message \mathbf{m} .

The codeword $\mathbf{X} = \text{Enc}(\mathbf{m}, \mathbf{R})$ where the randomness of encoding \mathbf{R} is a uniformly distributed r -bit string. Now consider an adversary with leakage variable \mathbf{Z} . Because of the one-to-one property of the regular encoder, we have

$$H_{\infty}(\mathbf{X}) = H_{\infty}(\mathbf{R}) = r, \quad (10)$$

and

$$\begin{aligned}\tilde{H}_\infty(\mathbf{X}|\mathbf{Z}) &= -\log \mathbb{E}_{\mathbf{Z}} (\max_{\mathbf{x}} \Pr[\mathbf{X} = \mathbf{x}|\mathbf{Z} = \mathbf{z}]) \\ &= -\log \mathbb{E}_{\mathbf{Z}} (\max_{\mathbf{r}} \Pr[\mathbf{R} = \mathbf{r}|\mathbf{Z} = \mathbf{z}]) \\ &= \tilde{H}_\infty(\mathbf{R}|\mathbf{Z}).\end{aligned}\tag{11}$$

For a leakage variable \mathbf{Z} , we consider two classes of adversaries denoted by \mathbb{A}_Z and \mathbb{B}_Z , depending on the conditions that they must satisfy, as follows: $\mathbb{A}_Z()$ is an adversary whose leakage variable must satisfy a lower bound on $\tilde{H}_\infty(\mathbf{R}|\mathbf{Z})$ and, $\mathbb{B}_Z()$ is an adversary whose leakage variable must satisfy a lower bound on $\tilde{H}_\infty(\mathbf{X}|\mathbf{Z})$. Both adversaries, when applied to a vector x , use their leakage variables to select an offset vector to be added to a codeword. That is $\mathbb{A}_Z(x) = x + \Delta_z$ where $\Delta_z \in F_q^n$ is chosen dependent on the leakage $\mathbf{Z} = z$. We have the same definition for $\mathbb{B}_Z(x) = x + \Delta_z$.

i. strong LLR-AMD code \Rightarrow strong ρ -AMD

Now consider a $(q^k, q^n, 2^r, \alpha, \delta)$ -strong LLR-AMD code \mathbf{C} with encoder and decoder pair, (Enc, Dec) . For an adversary \mathbb{A}_Z whose leakage variable satisfies $\tilde{H}_\infty(\mathbf{R}|\mathbf{Z}) \geq (1 - \alpha)r$, we have

$$\Pr[\text{Dec}(\mathbb{A}_Z(\text{Enc}(\mathbf{m}, \mathbf{R}))) \notin \{\mathbf{m}, \perp\}] \leq \delta,$$

where the probability is over the randomness of encoding, and is an expectation over $\mathbf{z} \in \mathcal{Z}$.

Note that using (10) and (11), the \mathbb{A}_Z adversary is also a \mathbb{B}_Z adversary satisfying,

$$\tilde{H}_\infty(\mathbf{X}|\mathbf{Z}) \geq \tilde{H}_\infty(\mathbf{X}) - \alpha r\tag{12}$$

Both these adversaries have the same leakage variable \mathbf{Z} and so any algorithm $\text{Offset}(z)$ used by one, taking the value $\mathbf{Z} = z$ as input and finding the the offset Δ_z , can be used by the other also (the two adversaries have the same information). This means that the success probabilities of the two adversaries are the same,

$$\Pr[\text{Dec}(\mathbb{A}_Z(\text{Enc}(\mathbf{m}, \mathbf{R}))) \notin \{\mathbf{m}, \perp\}] = \Pr[\text{Dec}(\mathbb{B}_Z(\text{Enc}(\mathbf{m}, \mathbf{R}))) \notin \{\mathbf{m}, \perp\}] \leq \delta.$$

For ρ -AMD codes, security is defined against a \mathbb{B}_Z adversary whose leakage variable \mathbf{Z} satisfies,

$$\tilde{H}_\infty(\mathbf{X}|\mathbf{Z}) \geq H_\infty(\mathbf{X}) - \rho n \log q\tag{13}$$

Comparing (13) and (12), we conclude that \mathbf{C} is a ρ -AMD code for ρ values that satisfy $\alpha r \geq \rho n \log q$, namely $\rho \leq \frac{\alpha r}{n \log q}$.

ii. strong ρ -AMD \Rightarrow strong LLR-AMD code

An argument similar to **i.** immediately gives that the $(q^k, q^n, 2^r, \alpha, \delta)$ -strong LLR-AMD code obtain from ρ -AMD code should satisfy $\alpha \leq \frac{\rho n \log q}{r}$. Next we

show the bound on r follows from Proposition 1 together with (10). Indeed, by Proposition 1, $\tilde{H}_\infty(\mathbf{X}|\mathbf{Z}) \geq \log \frac{1}{\delta}$ should hold for any \mathbf{Z} satisfying $\tilde{H}_\infty(\mathbf{X}|\mathbf{Z}) \geq H_\infty(\mathbf{X}) - \rho n \log q$. In particular, we must have $H_\infty(\mathbf{X}) - \rho n \log q \geq \log \frac{1}{\delta}$. Now we can use (10) to conclude that $r \geq \log \frac{1}{\delta} + \rho n \log q$. \square

B Proof of Lemma 5

Proof. The encoder Enc is a one-to-one correspondence between messages and codewords. Consider a message variable $\mathbf{M} \leftarrow \mathcal{M}$ (in particular, the uniform distribution is emphasized by $\mathbf{M}_u \stackrel{\$}{\leftarrow} \mathcal{M}$). The codeword is a variable $\mathbf{X} = \text{Enc}(\mathbf{M})$. Now consider an adversary with leakage variable \mathbf{Z} . Because of the one-to-one property of the encoder, we have

$$H_\infty(\mathbf{X}) = H_\infty(\mathbf{M}), \quad (14)$$

and

$$\begin{aligned} \tilde{H}_\infty(\mathbf{X}|\mathbf{Z}) &= -\log \mathbb{E}_{\mathbf{Z}} (\max_{\mathbf{x}} \Pr[\mathbf{X} = \mathbf{x}|\mathbf{Z} = \mathbf{z}]) \\ &= -\log \mathbb{E}_{\mathbf{Z}} (\max_{\mathbf{m}} \Pr[\mathbf{M} = \mathbf{m}|\mathbf{Z} = \mathbf{z}]) \\ &= \tilde{H}_\infty(\mathbf{M}|\mathbf{Z}). \end{aligned} \quad (15)$$

For a leakage variable \mathbf{Z} , we consider two classes of adversaries denoted by \mathbb{A}_Z and \mathbb{B}_Z , depending on the conditions that they must satisfy, as follows: $\mathbb{A}_Z()$ is an adversary whose leakage variable must satisfy a lower bound on $\tilde{H}_\infty(\mathbf{M}|\mathbf{Z})$ and, $\mathbb{B}_Z()$ is an adversary whose leakage variable must satisfy a lower bound on $\tilde{H}_\infty(\mathbf{X}|\mathbf{Z})$. Both adversaries, when applied to a vector x , use their leakage variables to select an offset vector to be added to a codeword. That is $\mathbb{A}_Z(x) = x + \Delta_z$ where $\Delta_z \in F_q^n$ is chosen dependent on the leakage $\mathbf{Z} = z$. We have the same definition for $\mathbb{B}_Z(x) = x + \Delta_z$.

i. weak LLR-AMD code \Rightarrow weak ρ -AMD

Now consider a $(q^k, q^n, \alpha, \delta)$ -weak LLR-AMD code \mathbf{C} with encoder and decoder pair, (Enc, Dec) . For an adversary \mathbb{A}_Z whose leakage variable satisfies $\tilde{H}_\infty(\mathbf{M}|\mathbf{Z}) \geq (1 - \alpha)k \log q$, we have

$$\Pr[\text{Dec}(\mathbb{A}_Z(\text{Enc}(\mathbf{M}))) \notin \{\mathbf{M}, \perp\}] \leq \delta,$$

where the probability is over the randomness of encoding, and is an expectation over $\mathbf{z} \in \mathcal{Z}$.

Note that using (14) and (15), the \mathbb{A}_Z adversary is also a \mathbb{B}_Z adversary satisfying,

$$\tilde{H}_\infty(\mathbf{X}|\mathbf{Z}) \geq (1 - \alpha)k \log q \quad (16)$$

Both these adversaries have the same leakage variable \mathbf{Z} and so any algorithm $\text{Offset}(z)$ used by one, taking the value $\mathbf{Z} = z$ as input and finding the the

offset Δ_z , can be used by the other also (the two adversaries have the same information). This means that the success probabilities of the two adversaries are the same,

$$\Pr[\text{Dec}(\mathbb{A}_Z(\text{Enc}(\mathbf{M}))) \notin \{\mathbf{M}, \perp\}] = \Pr[\text{Dec}(\mathbb{B}_Z(\text{Enc}(\mathbf{M}_u))) \notin \{\mathbf{M}_u, \perp\}] \leq \delta.$$

For ρ -AMD codes, security is defined against a \mathbb{B}_Z adversary whose leakage variable \mathbf{Z} satisfies,

$$\tilde{H}_\infty(\mathbf{X}|\mathbf{Z}) \geq H_\infty(\mathbf{X}) - \rho n \log q, \text{ where } \mathbf{X} = \text{Enc}(\mathbf{M}_u). \quad (17)$$

Comparing (17) and (16), we conclude that \mathbf{C} is a ρ -AMD code for ρ values that satisfy $\alpha k \geq \rho n$, namely $\rho \leq \frac{\alpha k}{n}$.

ii. weak ρ -AMD \Rightarrow weak LLR-AMD code

An argument similar to **i.** immediately gives that the $(q^k, q^n, \alpha, \delta)$ -weak LLR-AMD code obtain from ρ -AMD code should satisfy $\alpha \leq \frac{\rho n}{k}$. \square

C Proof of Proposition 2

Proof. By Proposition 1, $\tilde{H}_\infty(\mathbf{X}|\mathbf{Z}) \geq \log \frac{1}{\delta}$ should hold for any \mathbf{Z} satisfying $\tilde{H}_\infty(\mathbf{X}|\mathbf{Z}) \geq H_\infty(\mathbf{X}) - \rho n \log q$. In particular, we must have $H_\infty(\mathbf{X}) - \rho n \log q \geq \log \frac{1}{\delta}$. Since the message \mathbf{M} of weak ρ -AMD is uniform and the encoder is one-to-one correspondence, $H_\infty(\mathbf{X}) = H_\infty(\mathbf{M}) = k \log q$. We conclude that $k \log q - \rho n \log q \geq \log \frac{1}{\delta}$, namely,

$$q^{\rho n - k} \leq \delta. \quad (18)$$

Similar to the proof of Theorem 1, we also consider a random attack strategy. Then the total number of valid codewords that do not decode to \mathbf{M} is at least $(q^k - 1)$, which is the number of offsets that lead to undetected manipulations. A randomly chosen offset ($\Delta \neq 0^n$) leads to undetected manipulation with probability at most

$$\frac{q^k - 1}{q^n - 1}$$

and we must have

$$\frac{q^k - 1}{q^n - 1} \leq \delta. \quad (19)$$

\square

D Proof of Construction 4

Proof. Let β be a primitive element of \mathbb{F}_q . Then every element $m_i \in \mathbb{F}_q^*$ can be written as a power of β : $m_i = \beta^{m'_i}$. (9) is rewritten as follows.

$$f(\mathbf{m}, G) = \sum_{j=1}^k \beta^{\sum_{i=1}^k m'_i g_{i,j}} \bmod (q-1).$$

According to [2, Theorem 4] and the proof therein, (Enc, Dec) satisfies $\Pr[\text{Dec}(\text{Enc}(\mathbf{m}) + \Delta(\mathbf{Z}_\rho)) \notin \{\mathbf{m}, \perp\}] \leq \frac{\psi k}{q-1}$ as long as the leakage parameter ρ satisfies $k - (k+1)\rho \geq 1$. What is left to show is for any $\rho < 1$ and $\delta > 0$, there exists an N such that for all $k+1 \geq N$, $k - (k+1)\rho > 0$ and $\frac{\psi k}{q-1} \leq \delta$ are both satisfied. Indeed, $k - (k+1)\rho = k(1-\rho) - \rho$, which is bigger than 1 if $k > \frac{1+\rho}{1-\rho}$. So we can simply let $N = \lceil \frac{1+\rho}{1-\rho} \rceil + 1$. And $\frac{\psi k}{q-1} \leq \delta$ can be achieved by choosing a big enough q , for example, $q = \omega(\psi k)$ and choose a big enough k . \square